微分積分学第二（F1 クラス）試験問題解答

1 次の関数 $z = f(x,y)$ の偏導関数 z_x, z_y, z_{xy} の $(x,y) = (1,1)$ における値を求めよ．

(1) $z = \sqrt{4x - 5y^2 + 2}$

$$z_x = \frac{2}{\sqrt{4x - 5y^2 + 2}} = 2, \quad z_{xy} = \frac{10y}{(4x - 5y^2 + 2)^{3/2}} = 10$$

(2) $z = \log|-3x^2 + 2y^3|$

$$z_x = \frac{-6x}{-3x^2 + 2y^3} = 6, z_{xy} = \frac{36xy^2}{(-3x^2 + 2y^3)^2} = 36$$

2

(1)　関数 $z = (2x^2 - y^2)\sin(\pi xy/2)$ のグラフの $(x,y) = (1,1)$ における接平面の方程式を求めよ．

$$z = 1, \quad z_x = 4x\sin(\pi xy/2) + (2x^2 - y^2)(\pi y/2)\cos(\pi xy/2) = 4,$$

$$z_y = -2y\sin(\pi xy/2) + (2x^2 - y^2)(\pi x/2)\cos(\pi xy/2) = -2$$

$$z = 1 + 4(x-1) - 2(y-1) = 4x - 2y - 1$$

(2) 関数 $f(x,y) = y^2 e^{2y^2 - 3x^2}$ の $(x,y) = (2,1)$ を通る等高線を ℓ とする。ℓ をグラフとする陰関数を $y = \phi(x)$ とする．$y = \phi(x)$ の導関数 $dy/dx, d^2y/dx^2$ の，$x = 2$ における値をもとめよ．

$$2yy'e^{2y^2-3x^2} + y^2e^{2y^2-8x^2}(4yy' - 6x) = 0, \quad 2yy' + y^2(4yy' - 6x) = 0, \quad (1 + 2y^2)y' - 3xy = 0,$$

$$dy/dx = 3xy/(1 + 2y^2) = 6/3 = 2,$$

$$4yy'y' + (1 + 2y^2)y'' - 3y - 3xy' = 0,$$

$$y'' = (3y + 3xy' - 4yy'^2)/(1 + 2y^2) = -1/3$$

3 次の関数が極大値、極小値を取る点を調べ極値を求めよ．

(1) $f(x,y) = 3x^2 + xy + 2y^2 - 4x + 7y + 6$

(2) $f(x,y) = x^4 - 2x^2y^2 - 4y^4 + 4y^3$

(1) $f_x = 6x + y - 4 = 0, f_y = x + 4y + 7 = 0 \Rightarrow x = 1, y = -2$

$f_{xx} = 6, f_{xy}1, f_{yy} = 4, H = 24 - 1 = 23 > 0 f_{xx}(1,-2) = 6 > 0 \quad f(1,-2) = -3$　極小値

(2)　$f_x = 4x^3 - 4xy^2 = 4x(x^2 - y^2) = 4x(x-y)(x+y),$

$f_y = -4x^2y - 16y^3 + 12y^2 = -4y(x^2 + 4y^2 - 3y),$

$f_{xx} = 12x^2 - 4y^2, f_{yy} = -4x^2 - 48y^2 + 24y, f_{xy} = -8xy,$

$x=0 \Rightarrow y=0, \quad \text{or} \quad y=3/4,$

$x=y \Rightarrow y=0, x=0 \quad \text{or} \quad y=3/5, x=3/5$

$x=-y \Rightarrow y=0, x=0 \quad \text{or} \quad y=3/5, x=-3/5$

$H(0,3/4)=81/4>0, f_{xx}(0,3/4)=-9/4, f(0,3/4)=27/64$ 極大値

$H(3/5,3/5)=-2592/125<0, H(-3/5,3/5)=-2592/125<0,$ 極値をとらない

$f(0,y)=-4y^4+4y^3=4y^3(1-y)$

$0<y<1$ のとき $f(0,y)>0$, $-1<y<0$ のとき $f(0,y)<0$ であるから $(0,0)$ において極値をとらない.

4 次の積分を計算せよ.

(1) $\displaystyle\int\int_D 2xy\sin(xy^2)dxdy, \quad D:=\{(x,y): 0\le x\le \pi, 0\le y\le 1\}$

$$\begin{aligned}\int_D 2xy\sin(xy^2)dxdy &= \int_0^\pi\int_0^1 2xy\sin(xy^2)dydx \\ &= \int_0^\pi[-\cos(xy^2)]_{y=0}^{y=1}dx = \int_0^\pi(-\cos x+1)dx = \pi.\end{aligned}$$

(2) $\displaystyle\int\int_D 12(x-2y)^5dxdy \quad D:=\{(x,y): 0\le x\le y\le 1\}$

$$\begin{aligned}\int_D 12(x-2y)^5dxdy &= \int_0^1\int_0^y 12(x-2y)^5dxdy \\ &= \int_0^1[2(x-2y)^6]_{x=0}^{x=y}dy = \int_0^1(-126y^6)dy = -18\end{aligned}$$

(3) $\displaystyle\int\int_D(-x+2y)(2x+y)^2dxdy \quad D=\{(x,y): -1\le -x+2y\le 1, -1\le 2x+y\le 1\}$

$$u=-x+2y, v=2x+y \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 2 \\ 2 & 1\end{vmatrix} = -1-4=-5, \frac{\partial(x,y)}{\partial(u,v)} = -1/5$$

$$\int\int_D(-x+2y)(2x+y)^2dxdy = \int_{-1}^1\int_{-1}^1(uv^2)\frac{1}{5}dudv = [u^2/2]_{-1}^1[v^3/3]_{-1}^1\frac{1}{5} = 0$$

(4) $\displaystyle\int\int_D\frac{x+2y}{x^2+y^2+1}dxdy \quad D=\{(x,y): x^2+y^2\le 1, y\ge 0\}$

$$\begin{aligned}\int\int_D\frac{x+2y}{x^2+y^2+1}dxdy &= \int_0^\pi\int_0^1\frac{r\cos\theta+2r\sin\theta}{r^2+1}rdrd\theta = \int_0^\pi\int_0^1\frac{2r\sin\theta}{1+r^2}rdrd\theta \\ &= 2\int_0^\pi\sin\theta d\theta\int_0^1\frac{r^2}{1+r^2}dr = 2\times 2\times(1-\pi/4) = 4-\pi\end{aligned}$$

(5) $\displaystyle\int_C(-x^2y+3y^2)dx+6xydy \quad C: x=\cos\theta, y=\sin\theta, \quad 0\le\theta\le 2\pi$

$$\begin{aligned}\int_C(-x^2y+3y^2)dx+6xydy &= \int\int_D(x^2-6y+6y)dxdy = \int\int_D x^2dxdy \\ &= \int_0^{2\pi}\int_0^1 r^3\cos^2\theta drd\theta = \frac{1}{4}\int_0^{2\pi}\frac{1+\cos 2\theta}{2} = \frac{\pi}{4}\end{aligned}$$

微分積分学第二（E1 クラス）試験問題解答

1 次の関数 $z = f(x,y)$ の偏導関数 z_x, z_y, z_{xy} の $(x,y) = (1,1)$ における値を求めよ.

(1) $z = \sqrt{6x - 7y^2 + 2}$

$$z_x = \frac{3}{\sqrt{6x-7y^2+2}} = 3, z_{xy} = \frac{21y}{(6x-7y^2+2)^{3/2}} = 21$$

(2) $z = \log|2x^3 - 3y^2|$

$$z_x = \frac{6x^2}{2x^3-3y^2} = 6, z_y = \frac{-6y}{2x^3-3y^2} = -6, z_{xy} = \frac{36x^2y}{(2x^3-3y^2)^2} = 36$$

2

(1)　関数 $z = (x^2 - 2y^2)\sin(\pi xy/2)$ のグラフの $(x,y) = (1,1)$ における接平面の方程式を求めよ.

$$z = -1, \quad z_x = 2x\sin(\pi xy/2) + (x^2-2y^2)(\pi y/2)\cos(\pi xy/2) = 2,$$

$$z_y = -4y\sin(\pi xy/2) + (x^2-2y^2)(\pi x/2)\cos(\pi xy/2) = -4$$

$$z = -1 + 2(x-1) - 4(y-1) = 2x - 4y + 1$$

(2) 関数 $f(x,y) = x^2e^{2x^2-3y^2}$ の $(x,y) = (1,2)$ を通る等高線を ℓ とする。ℓ をグラフとする陰関数を $y = \psi(x)$ の形とするとき、その導関数 $dy/dx, d^2y/dx^2$ の $x = 1$ における値を求めよ。

$$2xe^{2x^2-yx^2} + x^2e^{2x^2-3y^2}(4x - 6ydy/dx) = 0,$$

$$2x + x^2(4x - 6ydy/dx) = 0, 1 + 2x^2 - 3xydy/dx = 0, dy/dx = (1+2x^2)/3xy = 3/6 = 1/2$$

$$4x - 3y'xy' - 3yy' - 3xyy'' = 0, y'' = (4x - 3xy'^2 - 3yy')/3xy = 1/24$$

3 次の関数が極大値、極小値を取る点を調べ極値を求めよ.

(1) $f(x,y) = 2x^2 - 3xy + 4y^2 - 11x + 14y + 15$

(2) $f(x,y) = x^4 - 2x^2y^2 - 4y^4 + 4y^3$

(1) $f_x = 4x - 3y - 11 = 0, f_y = -3x + 8y + 14 = 0 \Rightarrow x = 2, y = -1$

$f_{xx} = 4, f_{xy} = -3, f_{yy} = 8, H = 32 - 9 = 23 > 0 f_{xx}(2,-1) = 4 > 0 \quad f(2,-1) = -3$ 極小値

(2) $f_x = 4x^3 - 4xy^2 = 4x(x^2 - y^2) = 4x(x-y)(x+y),$

$f_y = -4x^2y - 16y^3 + 12y^2 = -4y(x^2 + 4y^2 - 3y),$

$f_{xx} = 12x^2 - 4y^2, f_{yy} = -4x^2 - 48y^2 + 24y, f_{xy} = -8xy,$

$x = 0 \Rightarrow y = 0, \quad \text{or} \quad y = 3/4,$

$x = y \Rightarrow y = 0, x = 0$ or $y = 3/5, x = 3/5$

$x = -y \Rightarrow y = 0, x = 0$ or $y = 3/5, x = -3/5$

$H(0, 3/4) = 81/4 > 0, f_{xx}(0, 3/4) = -9/4, f(0, 3/4) = 27/64$ 極大値

$H(3/5, 3/5) = -2592/125 < 0, H(-3/5, 3/5) = -2592/125 < 0,$

$f(0, y) = -4y^4 + 4y^3 = 4y^3(1 - y)$

$0 < y < 1$ のとき $f(0, y) > 0$, $-1 < y < 0$ のとき $f(0, y) < 0$ であるから $(0, 0)$ において極値をとらない.

4 次の積分を計算せよ.

(1) $\displaystyle\int\int_D 2xy\sin(xy^2)dxdy, \quad D := \{(x, y) : 0 \le x \le \pi, 0 \le y \le 1\}$

$$\begin{aligned}\int_D 2xy\sin(xy^2)dxdy &= \int_0^\pi\int_0^1 2xy\sin(xy^2)dydx \\ &= \int_0^\pi [-\cos(xy^2)]_{y=0}^{y=1}dx = \int_0^\pi(-\cos x + 1)dx = \pi.\end{aligned}$$

(2) $\displaystyle\int\int_D 12(x - 2y)^5 dxdy \quad D := \{(x, y) : 0 \le x \le y \le 1\}$

$$\begin{aligned}\int_D 12(x-2y)^5 dxdy &= \int_0^1\int_0^y 12(x-2y)^5 dxdy \\ &= \int_0^1 [2(x-2y)^6]_{x=0}^{x=y}dy = \int_0^1(-126y^6)dy = -18\end{aligned}$$

(3) $\displaystyle\int\int_D (-x + 2y)(2x + y)^2 dxdy \quad D = \{(x, y) : -1 \le -x + 2y \le 1, -1 \le 2x + y \le 1\}$

$$u = -x + 2y, v = 2x + y \Rightarrow \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -1 - 4 = -5, \frac{\partial(x, y)}{\partial(u, v)} = -1/5$$

$$\int\int_D(-x+2y)(2x+y)^2dxdy = \int_{-1}^1\int_{-1}^1(uv^2)\frac{1}{5}dudv = [u^2/2]_{-1}^1[v^3/3]_{-1}^1\frac{1}{5} = 0$$

(4) $\displaystyle\int\int_D \frac{x + 2y}{x^2 + y^2 + 1}dxdy \quad D = \{(x, y) : x^2 + y^2 \le 1, y \ge 0\}$

$$\begin{aligned}\int\int_D \frac{x+2y}{x^2+y^2+1}dxdy &= \int_0^\pi\int_0^1 \frac{r\cos\theta + 2r\sin\theta}{r^2+1}rdrd\theta = \int_0^\pi\int_0^1\frac{2r\sin\theta}{1+r^2}rdrd\theta \\ &= 2\int_0^\pi \sin\theta d\theta\int_0^1\frac{r^2}{1+r^2}dr = 2\times 2\times(1 - \pi/4) = 4 - \pi\end{aligned}$$

(5) $\displaystyle\int_C (-x^2y + 3y^2)dx + 6xydy \quad C : x = \cos\theta, y = \sin\theta, \quad 0 \le \theta \le 2\pi$

$$\begin{aligned}\int_C(-x^2y+3y^2)dx + 6xydy &= \int\int_D(x^2 - 6y + 6y)dxdy = \int\int_D x^2dxdy \\ &= \int_0^{2\pi}\int_0^1 r^3\cos^2\theta drd\theta = \frac{1}{4}\int_0^{2\pi}\frac{1+\cos 2\theta}{2} = \frac{\pi}{4}\end{aligned}$$